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Model documentation

Group 2

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# **Executive Summary**

This is a review of Group 2 risk model for a portfolio of a long and short position of Ford stocks, Xerox Stocks and S&P 500 At-The-Money Option Contract. The risk model is used to calculate Value-at-Risk (VaR), and Expected Shortfall (ES). Value-at-Risk (VaR) is a numerical result—over which there is a certain percentage of probability that the losses could exceed. Expected Shortfall (ES), on the other hand, defines the average of the losses conditional on losses that are greater than VaR.

Pricing model has two components. The first one is used to calculate its price, and the second one is used to measure risks. In the end, backtesting will be applied to justify model validation.

Since option has a nonlinear payoff, it is impossible to calculate the parametric VaR using the variance-covariance matrix under the normal distribution. For the convenience of calculation, Parametric VaR in this model will take into consideration the portfolio under the Geometric Brownian Motion (GBM) process to make the calculation.

This model will also calculate Monte Carlo VaR and ES, Historical VaR and ES, and Parametric VaR and ES. It uses a historical time period to apply to the current market positions and repricing. ES has a more coherent result than VaR since it meets four of the following requirements:

* Monotonicity: if a portfolio produces a worse result than another portfolio for every state of the world, its risk measure should be greater.
* Translation Invariance: if an amount of cash K is added to a portfolio, its risk measure should go down by K.
* Homogeneity: changing the size of a portfolio by a factor lambda, while keeping the relative amounts of different items in the portfolio items in the portfolio the same, should result in the risk measure being multiplied by lambda.
* Subadditivity: The risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

However, ES calculation is not elicitable and thus increases some difficulties in backtesting.

The strengths of our model are, that our model applies industry standards model to better calculate risks, and that we successfully validate the model by a robust back-testing plan.

The risk of this model is the uncertainty in change of the correlation between each stock and each risk factors. If we underestimated the correlation, the total risk could be undermeasured and thus underhedged as well.

# **Introduction**

This is a review of Group 2 risk model for a portfolio of stocks including Ford and Xerox and S&P 500 Index Option. The main purpose of this model is to calculate the price of the portfolio and measure market risk.

Risk calculations include two major components. The first major part is pricing model for calculation which involves Black-Scholes Merton model for the option pricing, and the second part is the model for risk measurement to Value-at-Risk (VaR) and Expected Shortfall (ES). Finally, there will be a backtesting process to justify the validity of the VaR model. All components will be documented.

# **Product Description**

This product contains two important instruments, one for stock, and one for option.

US Stocks are traded upon New York Stock Exchange (NYSE), and documentation for those companies are filed under Security and Exchange Committee (SEC). This model picks Ford’s stock and Xerox’s stock into our portfolio.

Option is a derivative that provides a right but not an obligation to execute. Call options are the right of execution that one can purchase a stock with a predefined price (strike price) that could be lower than a stock price at maturity of the contract, and Put options are the right of execution that one can sell a stock with a predefined price (strike price) that could be higher than a stock price at maturity of the contract. The options are traded in Chicago Board Options Exchange (CBOE) and American Stock Exchange (AMEX). In this model, we presume that option traded are European options which can only executed at the maturity.

Payoffs of a call option and a put option are:

In this model, S&P 500 Index Option was chosen. Option contracts are all At-The-Money (that is, the spot price equals to the strike price), with its Time-to-Maturity as one year (12 month). In addition, it is presumed that implied volatility of an option contract is a flat surface which is insensitive to time-to-maturity and strike price.

Data have been already downloaded whose latest date is 8/30/2019. Time when the portfolio was purchased is 09/05/1997.

# **Pricing Model Description**

Pricing Model applies Black-Scholes formula in Option pricing calculation. Black-Scholes formula assumes that a non-dividend stock price follows a Geometric Brownian Motion, given the initial condition of the stock price :

where µ is the drift of the stock price, and σ is the volatility of the stock price.

The Black-Scholes-Merton Model implies a risk-neutral, no-arbitrage market under a constant risk-free rate r, with assumptions of a stock price following a Geometric Brownian Motion. Black and Scholes derived a Partial Differential Equations for the solution of the option: Suppose V(S(t),t) is the option, and S(t) is the stock price, by applying Ito’s lemma, the whole differentiation becomes:

Under Risk-neutral Distribution, , which is the Black-Scholes Partial Differential Equation The boundary condition of the PDE is the payoff of the options.

Another perspective is the usage of a numeraire. A money market account numeraire B is specified under risk-neutral distribution, given the initial condition of :

Applying Ito’s lemma, this could be rewritten as:

That is, , and

Hence, the pricing formula of Option, denoted as V, is calculated under a risk-neutral distribution Q if risk-free return is non-stochastic:

Where

Hence, price for a call option is:

And price for a put option is:

Where is the Standard Normal Cumulative Density Function.

## **Pricing Model Calibration**

The pricing model inputs are:

* S&P 500 Index Spot Price
* S&P 500 Index option Strike Price
* Ford Stock Price
* Xerox Stock Price
* 12-month S&P 500 At-The-Money Implied Volatility

To calibrate the most fitted parameters—drift and volatility—into the Geometric Brownian Motion process, this model first automatically calculates daily log-returns for the index and stock prices, then using a methodology of rolling window with N size to generate a rolling mean and standard deviation for the log-returns. Geometric Brownian Motion process indicates that:

, where

Approximately, , and . Besides, U.S. has 252 transaction days for a year, and thus . The calibration of parameters for drift and volatility suggests that:

In addition, two different log-return also incur correlation , which can be calibrated by:

, where is daily correlation using a rolling window N.

## **Pricing Model Implementation**

In this model, a 12-month S&P 500 At-The-Money implied volatility have already downloaded from the Bloomberg Terminal. However, this documentation will discuss the algorithm of deriving the implied volatility.

Implied volatility surface is not observable; instead, Black-Scholes formula is applied to estimate from spot price, strike price, Interest rate, time to maturity, and Mart-To-Market price in the market.

One of approaches to search implied volatility is Newton’s method. Newton’s model assumes that there is a function that is continuous and differentiable within an interval. Newton’s method suggests that such computation could be conducted to find a root that usually converges:

1. Using Taylor expansion, a linear approximation should be established:
2. Assume that , and apply linear approximation to solve for x:
3. Reiterate this process for n times, until relative error is smaller than tolerate (this model sets default tolerance as 1e-4):

As such, a function was established to solve for the root:

Where is Vega of the option when sigma is computed n times. In this model, it already presumed that the implied volatility is flat, that is irresponsive to the change in time-to-maturity and strike price.

# **Risk Measurement Model Description**

The second major part of this model is risk measurement. Market now utilizes the Value-at-Risk and Expected Shortfall to illustrate the severity of the risk.

Value-at-Risk (VaR) is a numerical result—over which there is a certain percentage of probability that the losses could exceed. Given the distribution of Profit and Loss (P&L) and a percentage *p*, VaR can be defined by . Thus, , where *F* is the P&L Cumulative Density Function.

Expected Shortfall (ES), on the other hand, defines the average of the losses conditional on losses that are greater than VaR; that is, .

ES calculation is more coherent than VaR since it meets all of four following requirements:

* Monotonicity: if a portfolio produces a worse result than another portfolio for every state of the world, its risk measure should be greater.
* Translation Invariance: if an amount of cash K is added to a portfolio, its risk measure should go down by K.
* Homogeneity: changing the size of a portfolio by a factor lambda, while keeping the relative amounts of different items in the portfolio items in the portfolio the same, should result in the risk measure being multiplied by lambda.
* Subadditivity: The risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

However, ES is not elicitable so that back-testing may have some difficulties.

This model assumes the initial position is $10,000, and that a user exhausts the initial position. For the whole portfolio, the value at time T is no more than the sum of the investment in the stock and the option:

This risk measurement model calculates Monte-Carlo VaR and ES, historical VaR and ES, and Parametric VaR and ES respectively.

## **Historical VaR and Historical ES**

Historical VaR simply leverages 5-days actual returns of portfolio over a rolling window N to determine its 99% percentile after sorting. Historical ES takes average of losses exceeding 97.5% Historical VaR.

## **Parametric VaR and ES**

Traditionally, Parametric VaR assumes that the portfolio returns as a whole will follow a normal distribution to generate a 99% percentile after sorting. Suppose there are N instruments in the portfolio, with weights assigned to those instruments, which is denoted as a vector . Based on returns vector , and variance-covariance matrix , it is easy to determine the normal distribution of the portfolio P:

Hence, 99% VaR is determined by

However, since option price yields a nonlinear payoff, it is impossible to apply variance-covariance matrix to calculate the result if there is an option in the portfolio. Hence, this model will not conduct the traditional parametric VaR calculation of the portfolio in this case. It will instead calculate the VaR assuming that the whole portfolio follows a GBM process:

Where Π stands for the portfolio, µ for the drift of the portfolio’s return and σ for the volatility of the portfolio’s return.

In the model, the initial position sets a default value as $10,000. If the portfolio is in a long position, VaR and ES are given as follows:

If the portfolio is in a short position, VaR and ES are given as follows:

## **Monte Carlo VaR and Monte Carlo ES**

This model will determine Monte Carlo VaR and Monte Carlo ES by taking into consideration the portfolio itself. Monte Carlo simulation, presuming GBM process, leverages Weiner Process to generate 10000 different paths that appears in log-return of the whole portfolio whose distribution follows:

Where µ is the drift of the portfolio’s return, and σ is the volatility of the portfolio’s return, as t is the days in Monte-Carlo simulation, the default value of which is 5 days. As such, a 10000 path for the portfolio are generated on each day in the observation, and the portfolio P incurs a price at time t under Monte Carlo Simulation as a result of 10000 paths.

Next, this model will automatically calculate and sort the result of P&L over 10000 paths, and choose its 99% percentile through Monte Carlo Simulation as the Monte Carlo VaR. Monte Carlo ES takes average of 10000 losses conditional on exceeding 97.5% Monte Carlo VaR.

# **Model Validation**

The scope of Model validation involves VaR calculation. To validate the VaR, this model uses the Backtesting methodology, suggesting that it needs to calculate the real losses incurred in the past and count in a one-year rolling window exception that exceeds the VaR that the model implies.

In this backtesting process, we develop a hypothesis test:

The exception, denoted as X, should follow a binomial distribution; nevertheless, if the sample size is greater enough, a normal approximation can be applied: , where n is the number of the sample size in the rolling window (252 in a year), and p is the percentage of the exception (1%) in this case. Confidence interval under (100-α%) is given as

BASEL suggests a 95% confidence level two-tail test against 99% 5-day VaR will be conducted. Hence, confidence interval would be ; that is, we will reject the null hypothesis that this model is correct if the exception is greater than 6. BASEL Accord suggests that the exceptions be lower than five which indicates a Green zone, and thus there will not be punishment for the capital requirement.

Moreover, Fundamental Review of Trading Book (FRTB) suggests that the model will be rejected if the exception of 12 months over 99% VaR is more than 12 times.

# **Conclusion**

This model has one of the most disadvantages in an extreme case where there are only options in the model. Monte Carlo simulation simplifies the case by considering the portfolio as a whole to generate a GBM process. Nonetheless, it is not the Option but the Underlying stock that follows a GBM process, as a result of which the risk may overwhelmingly underestimated.

This model has the weakness where it may underestimate the correlation if the worse case scenario recurs. Therefore, a stress testing must be conducted choosing according to Dodd-Frank Act Stress Testing (DFAST) scenarios to simulate a worse case as to determine the level of VaR.

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